

Finite-dimensional noncommutative geometry

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Summary of this talk

1. Spectral geometry
2. Spectrally truncated geometry
3. Weyl laws
4. Quantum ergodicity

This talk is based on joint work with Oliver Fürst (Bonn), Malte Leimbach (Bonn) and Edward McDonald (Bonn/Paris).

Part 1: Spectral geometry

Spectral geometry

Can one hear the shape of a drum?



Figure 1: Mark Kac, Center for Nonlinear Studies.

Weyl's law (Weyl 1910s, Duistermaat–Guillemin 1975, Ivrii 1980)

Let (M, g) be a compact Riemannian manifold of dimension d satisfying some assumptions. Let $N(t)$ be the number of eigenvalues (counting multiplicities) of Δ_g with absolute value less than t . Then as $t \rightarrow \infty$,

$$N(t) = \frac{\omega_d}{(2\pi)^d} \text{Vol}(M)t^{\frac{d}{2}} - \frac{\omega_{d-1}}{4(2\pi)^{d-1}} \text{Area}(\partial M)t^{\frac{d-1}{2}} + o(t^{\frac{d-1}{2}}).$$

What we can hear 2

Heat trace expansion (Minakshisundaram–Pleijel 1949)

We have

$$\mathrm{Tr}(\exp(t\Delta_g)) \sim \sum_{k=0}^{\infty} a_k(\Delta_g) t^{\frac{k-d}{2}}, \quad t \rightarrow 0,$$

the first coefficients of which can be given as

$$a_0(\Delta_g) = (4\pi)^{-\frac{d}{2}} \mathrm{Vol}(M);$$
$$a_2(\Delta_g) = -\frac{1}{6}(4\pi)^{-\frac{d}{2}} \int_M R \, d\nu_g,$$

where R is the scalar curvature of M . The coefficients $a_k(\Delta_g)$ vanish for odd k , and the higher coefficients $a_k(\Delta_g)$ are integrals over M of (complicated) expressions involving the metric of M .

What we cannot hear

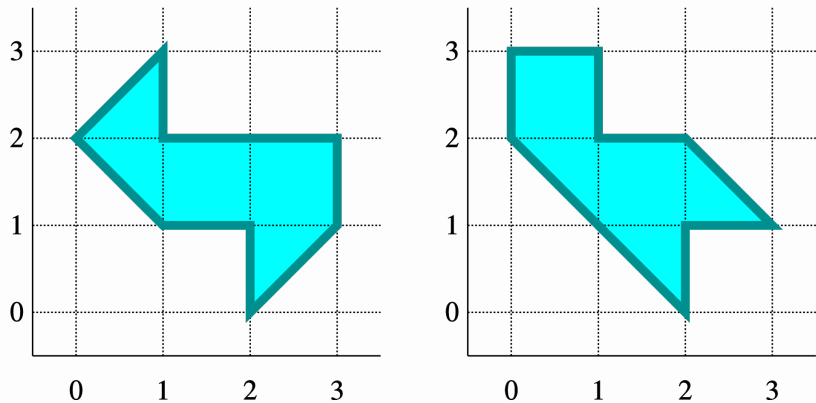


Figure 2: Isospectral drums.

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This changes if we add $C(M)$ to the mix, which contains the topology of M .

A spectral invariant

The data $(C(M), L_2(M), \Delta_g)$ is an invariant for M .

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Theorem (follows from Arendt–Biegrt–ter Elst 2012)

Let (M_1, g_1) and (M_2, g_2) be connected compact Riemannian manifolds, with corresponding Laplace–Beltrami operators Δ_1 and Δ_2 . Then the following are equivalent:

1. the Riemannian manifolds (M_1, g_1) and (M_2, g_2) are isometric;
2. there exists a unital $*$ -isomorphism $\psi : C(M_1) \xrightarrow{\sim} C(M_2)$ and a unitary operator $U : L_2(M_1) \rightarrow L_2(M_2)$ such that

$$UM_f = M_{\psi(f)}U, \quad f \in C(M_1)$$

$$U\Delta_1 = \Delta_2U.$$

Reconstruction theorem

We can do even better if we replace $C(M)$ by $C^\infty(M)$, and replace Δ_g by its square root D_S , a Dirac operator.

Connes' reconstruction theorem (2013)

Let $(\mathcal{A}, \mathcal{H}, D)$ be such that:

1. \mathcal{H} is a Hilbert space;
2. \mathcal{A} is a commutative $*$ -algebra represented as bounded operators on \mathcal{H} ;
3. D is a self-adjoint operator on \mathcal{H} with compact resolvent;
4. $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$;
5. a couple more technical assumptions.

Then we can construct $S \rightarrow M$ such that

$(\mathcal{A}, \mathcal{H}, D) \simeq (C^\infty(M), L_2(M, S), D_S)$, where D_S is the Dirac operator on the spinor bundle $S \rightarrow M$.

It therefore makes sense to interpret *spectral triples* $(\mathcal{A}, \mathcal{H}, D)$ as noncommutative spaces.

Definition (spectral triple)

A spectral triple is a triple $(\mathcal{A}, \mathcal{H}, D)$ such that:

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Connes' distance formula

For a manifold (M, g) ,

$$d_g(x, y) = \sup_{f \in C^\infty(M)} \{|f(x) - f(y)| : |\nabla f| \leq 1\}.$$

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Noting that $f \mapsto f(x)$ is a *state* on $C^\infty(M)$ (positive linear functional of norm 1), we can naturally this definition to all states $\mathcal{S}(C^\infty(M))$:

$$d(\phi, \psi) := \sup_{f \in C^\infty(M)} \{|\phi(f) - \psi(f)| : |\nabla f| \leq 1\}, \quad \phi, \psi \in \mathcal{S}(C^\infty(M)).$$

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Finally, Connes extended this distance to spectral triples $(\mathcal{A}, \mathcal{H}, D)$:

$$d_D(\phi, \psi) := \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}, \quad \phi, \psi \in \mathcal{S}(\mathcal{A}).$$

Part 2: Spectrally truncated geometry

Even if we regard Connes' reconstruction theorem as an answer to Kac's question, many questions remain.

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Realistically, we will only ever be able to access a finite-dimensional truncation of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. How much geometry is retained?

Taking a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a projection P , we can smash everything down to $(P\mathcal{A}P, P\mathcal{H}, PD)$. For example, by taking $P := \chi_I(D)$ a finite-rank spectral projection. (First studies by D'Andrea–Lizzi–Martinetti, comprehensive framework by Connes–van Suijlekom.)

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Note that $P\mathcal{A}P$ is no longer an algebra: it is in general not closed under products. Instead, it is an *operator system*.

Truncated Spectral Triples

Taking a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a projection P , we can smash everything down to $(P\mathcal{A}P, P\mathcal{H}, PD)$. For example, by taking $P := \chi_I(D)$ a finite-rank spectral projection. (First studies by D'Andrea–Lizzi–Martinetti, comprehensive framework by Connes–van Suijlekom.)

Note that $P\mathcal{A}P$ is no longer an algebra: it is in general not closed under products. Instead, it is an *operator system*.

The game is now to see what properties of $(\mathcal{A}, \mathcal{H}, D)$ can be recovered from the operator system spectral triple $(P\mathcal{A}P, P\mathcal{H}, PD)$ as $P \uparrow 1$.

Truncated state spaces

An operator system is a $*$ -closed unital subspace of $B(\mathcal{H})$. As such, we can again define the state space as the positive linear functionals of norm 1.

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Hence, for $(P\mathcal{A}P, P\mathcal{H}, PD)$, we can put an extended metric on $\mathcal{S}(P\mathcal{A}P)$ by setting

$$d_{PD}(\phi, \psi) := \sup_{T \in P\mathcal{A}P} \{|\phi(T) - \psi(T)| : \|[PD, T]\| \leq 1\}, \quad \phi, \psi \in \mathcal{S}(P\mathcal{A}P).$$

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We thus have two (extended) metric spaces $(\mathcal{S}(\mathcal{A}), d_D)$ and $(\mathcal{S}(P\mathcal{A}P), d_{PD})$. When do we have (Gromov–Hausdorff?) convergence as $P \uparrow 1$?

Part 3: Weyl laws

Microlocal Weyl law

Weyl's law gives for a compact Riemannian manifold (M, g) , where

$$P_\lambda = \chi_{[0, \lambda]}(\Delta),$$

$$\mathrm{Tr}(P_\lambda) \sim C_d \mathrm{vol}(M) \lambda^{\frac{d}{2}}, \quad \lambda \rightarrow \infty.$$

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There exists a *local* version of Weyl's law, which gives for $f \in C^\infty(M)$,

$$\mathrm{Tr}(P_\lambda M_f P_\lambda) = \sum_{n=0}^{N(\lambda)} \langle e_n, M_f e_n \rangle \sim C_d \lambda^{\frac{d}{2}} \int_M f d\nu_g, \quad \lambda \rightarrow \infty.$$

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Or, even, a *microlocal* Weyl law, which states for $A \in \Psi^0(M)$,

$$\mathrm{Tr}(P_\lambda A P_\lambda) = \sum_{n=0}^{N(\lambda)} \langle e_n, A e_n \rangle \sim C_d \lambda^{\frac{d}{2}} \int_{S^*M} \sigma_0(A) d\mu, \quad \lambda \rightarrow \infty.$$

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Combined, we obtain

$$\lim_{\lambda \rightarrow \infty} \frac{\mathrm{Tr}(P_\lambda A P_\lambda)}{\mathrm{Tr}(P_\lambda)} = C_d \int_{S^*M} \sigma_0(A) d\mu.$$

Dixmier traces

Let \mathcal{H} be a Hilbert space. An eigenvalue sequence of a compact operator $A \in K(\mathcal{H})$ is a sequence $\{\lambda(k, A)\}_{k \in \mathbb{N}}$ of the eigenvalues of A listed with multiplicity, such that $\{|\lambda(k, A)|\}_{k \in \mathbb{N}}$ is non-increasing.

The usual operator trace Tr can be characterised for trace class operators $A \in \mathcal{L}_1 \subset K(\mathcal{H})$ as

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The Dixmier trace is defined on so-called weak trace class operators $A \in \mathcal{L}_{1,\infty} \subset K(\mathcal{H})$ by

$$\text{Tr}_\omega(A) = \omega\text{-}\lim_{n \rightarrow \infty} \frac{1}{\log(2+n)} \sum_{k=1}^n \lambda(k, A),$$

where $\omega \in \ell_\infty(\mathbb{N})^*$ is an extended limit. Note that $\mathcal{L}_1 \subset \mathcal{L}_{1,\infty}$, but if $A \in \mathcal{L}_1$, $\text{Tr}_\omega(A) = 0$.

Connes' integral formula

Connes proved the following.

Connes' Integral Formula

Let (M, g) be a compact Riemannian manifold, $f \in C^\infty(M)$. Then for any Dixmier trace Tr_ω ,

$$\text{Tr}_\omega(M_f(1 - \Delta_g)^{-\frac{d}{2}}) = C_d \int_M f d\nu_g.$$

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Or stronger, for $A \in \Psi_{cl}^0(M)$,

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Connes' result is in fact even stronger, as he does not assume a Riemannian structure.

Comparison

Compare the microlocal Weyl law

$$\frac{\mathrm{Tr}(P_\lambda A P_\lambda)}{\mathrm{Tr}(P_\lambda)} \xrightarrow{\lambda \rightarrow \infty} \frac{1}{\mathrm{vol}(S^*M)} \int_{S^*M} \sigma_0(A) d\mu,$$

with Connes' formula

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$$\mathrm{Tr}_\omega(A(1 - \Delta)^{-\frac{d}{2}}) = C_d \int_{S^*M} \sigma_0(A) d\mu.$$

H.-McDonald

Let \mathcal{H} be a separable Hilbert space, $A \in B(\mathcal{H})$, D self-adjoint with compact resolvent, $P_\lambda := \chi_{[-\lambda, \lambda]}(D)$. If D satisfies Weyl's law $\lambda(k, |D|) \sim Ck^{\frac{1}{d}}$, then for all Dixmier traces Tr_ω ,

$$\frac{\mathrm{Tr}_\omega(A(1 + D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1 + D^2)^{-\frac{d}{2}})} = \omega \circ M \left(\frac{\mathrm{Tr}(P_{\lambda_n} A P_{\lambda_n})}{\mathrm{Tr}(P_{\lambda_n})} \right).$$

Here, $M : \ell_\infty \rightarrow \ell_\infty$ is the logarithmic averaging defined by

$$M(x)_n = \frac{1}{\log(n+2)} \sum_{k=0}^n \frac{x_k}{k}.$$

Hence, if $(\mathcal{A}, \mathcal{H}, D)$ is d -dimensional and D satisfies Weyl's law, then

$$P_\lambda A P_\lambda \mapsto \frac{\mathrm{Tr}(P_\lambda A P_\lambda)}{\mathrm{Tr}(P_\lambda)}$$

is a reasonable approximation of the noncommutative integral $\mathrm{Tr}_\omega(A(1 + D^2)^{-\frac{d}{2}})$.

Part 4: Quantum Ergodicity

A one-particle system described by an operator H on $L_2(M)$ is called “quantum ergodic” if the high energy states of H are ‘smeared’ over M . The field of Quantum Ergodicity studies when this is the case.

Picture

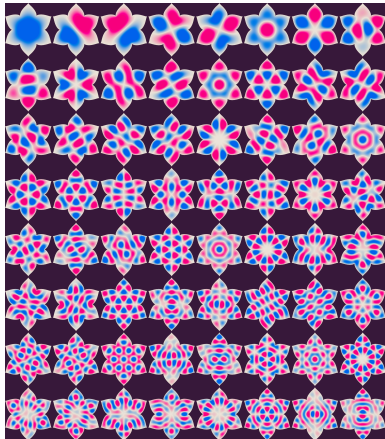


Figure 3: Eigenfunctions of the Laplacian on a rose-shaped domain, quantum ergodicity **unknown**.

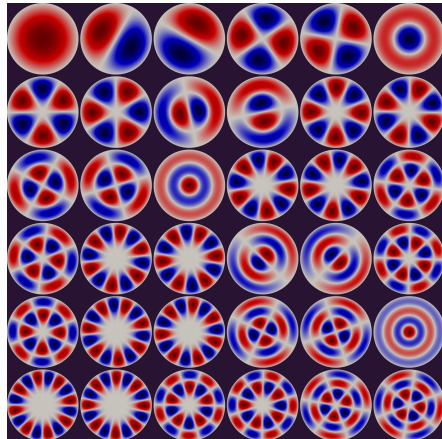


Figure 4: Eigenfunctions of the Laplacian on the disc, **not** quantum ergodic.

Quantum Ergodicity

For a compact Riemannian manifold M and a positive self-adjoint operator Δ on $L_2(M)$ with compact resolvent, Δ is said to be quantum ergodic if for every orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of $L_2(M)$ consisting of eigenfunctions of Δ , there exists a density one subsequence $J \subseteq \mathbb{N}$ such that

$$\lim_{J \ni j \rightarrow \infty} \langle e_j, A e_j \rangle_{L_2(M)} \rightarrow \frac{1}{\text{vol}(S^*M)} \int_{S^*M} \sigma_0(A) d\mu, \quad A \in \Psi^0(M),$$

where ν is a probability measure on M . In this context, a density one subsequence means that

$$\frac{\#(J \cap \{0, \dots, n\})}{n+1} \rightarrow 1, \quad n \rightarrow \infty.$$

QE as a Weyl law

We can interpret Quantum Ergodicity as a stronger microlocal Weyl law. Namely, the QE property

$$\lim_{J \ni j \rightarrow \infty} \langle e_j, Ae_j \rangle_{L_2(M)} \rightarrow \frac{1}{\text{vol}(S^*M)} \int_{S^*M} \sigma_0(A) d\mu, \quad A \in \Psi^0(M),$$

is equivalent by the Koopman–von Neumann lemma to

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=0}^N \left| \langle e_n, Ae_n \rangle - \frac{1}{\text{vol}(S^*M)} \int_{S^*M} \sigma_0(A) d\mu \right| = 0, \quad A \in \Psi^0(M).$$

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This is now recognisable as a stronger version of the microlocal Weyl law

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=0}^N \langle e_n, A e_n \rangle - \frac{1}{\text{vol}(S^*M)} \int_{S^*M} \sigma_0(A) d\mu = 0, \quad A \in \Psi^0(M).$$

Let (M, g) be a Riemannian manifold, and denote by $SM \subseteq TM$ the tangent vectors of length 1. Then we define the geodesic flow

$$G_t : SM \rightarrow SM, \quad t \in \mathbb{R},$$

in the usual way. By duality, we can likewise define $G_t : S^*M \rightarrow S^*M$, where $S^*M \subseteq T^*M$.

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The geodesic flow is said to be *ergodic* if for every measurable function $f \in L_\infty(S^*M)$ which is fixed by the flow (i.e. $f \circ G_t = f$ almost everywhere), it must be that f is constant almost everywhere.

Fundamental theorem of QE

The fundamental theorem that started the field of Quantum Ergodicity is the following.

Theorem (Shnirelman 1974, Zelditch 1987, Colin de Verdière 1985)

Let M be a compact Riemannian manifold. If the geodesic flow on M is ergodic, then the Laplace–Beltrami operator Δ_g is quantum ergodic.

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Let M be a compact Riemannian manifold. If the geodesic flow on M is ergodic, then the Laplace–Beltrami operator Δ_g is quantum ergodic.

By now, various extensions of this theorem exist. The common thread is to study geodesic flow, and translate this into asymptotic behaviour of eigenfunctions of an operator.

Noncommutative geodesic flow

Connes defined geodesic flow on spectral triples in 1996.

Noncommutative cosphere bundle

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple. Let $\sigma_t : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ be defined by $A \mapsto e^{it|D|} A e^{-it|D|}$. Then define the C^* -algebra

$$S^* \mathcal{A} := C^* \left(\bigcup_{t \in \mathbb{R}} \sigma_t(\mathcal{A}) + K(\mathcal{H}) \right) / K(\mathcal{H}),$$

where σ_t descends to an action of \mathbb{R} on $S^* \mathcal{A}$. If $(\mathcal{A}, \mathcal{H}, D) \simeq (C^\infty(M), L_2(S), D_S)$, then $S^* \mathcal{A} \simeq C(S^*M)$ and σ_t is the geodesic flow.

Since ergodicity of the geodesic flow is a measure theoretic statement, we need to take one more step.

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L_2 -cosphere bundle

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple where D satisfies Weyl's law. Then

$$\tau(A + K(\mathcal{H})) = \frac{\mathrm{Tr}_\omega(A(1 + D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1 + D^2)^{-\frac{d}{2}})}, \quad A + K(\mathcal{H}) \in S^*\mathcal{A},$$

defines a finite positive trace on $S^*\mathcal{A}$. Then define $L_2(S^*\mathcal{A})$ as the Hilbert space of the GNS representation of $S^*\mathcal{A}$ corresponding to τ .

The geodesic flow σ_t on $S^*\mathcal{A}$ descends to a unitary operator on $L_2(S^*\mathcal{A})$.

We can now naively put forward a definition of ergodic geodesic flow for spectral triples. Namely, we say that the geodesic flow σ_t is ergodic on $(\mathcal{A}, \mathcal{H}, D)$ if the only σ_t -invariant element of $L_2(S^*\mathcal{A})$ is the identity.

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NCG QE (H.-McDonald)

Let $(\mathcal{A}, \mathcal{H}, D)$ be a d -summable regular spectral triple where D satisfies Weyl's law, and with local Weyl laws. If the geodesic flow on $(\mathcal{A}, \mathcal{H}, D)$ is ergodic, then D is quantum ergodic. That is, for every basis $\{e_n\}_{n=0}^\infty$ of \mathcal{H} consisting of eigenvectors of D , there exists a density one subset $J \subseteq \mathbb{N}$ such that

$$\lim_{J \ni j \rightarrow \infty} \langle e_j, ae_j \rangle = \frac{\mathrm{Tr}_\omega(a(1 + D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1 + D^2)^{-\frac{d}{2}})}, \quad a \in \mathcal{A}.$$

Thanks for listening!