Connes' Integral Formula from 1915

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Summary of this talk

1. 1915: Szegő's limit theorem
2. 1979: Widom's argument
3. 1988: Connes' integral formula
4. 2025 (I): Noncommutative Szegő limit theorem
5. 2025 (II): NCG and Quantum Ergodicity

This talk is based on joint work with Ed McDonald (Université Paris-Est Créteil).

The Protagonists

Szegő Limit Theorem (1915)

Let $0 < w \in C(\mathbb{S}^1)$, and define the Toeplitz matrices

$$T_n := \begin{pmatrix} w_0 & w_1 & w_2 & \cdots & w_n \\ w_{-1} & w_0 & w_1 & \cdots & w_{n-1} \\ w_{-2} & w_{-1} & w_0 & \cdots & w_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{-n} & w_{-n+1} & w_{-n+2} & \cdots & w_0 \end{pmatrix},$$

where w_j are Fourier coefficients of w. Then,

$$\lim_{n\to\infty} (\det T_n)^{\frac{1}{n}} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log(w(\theta))\, d\theta\right).$$

The Protagonists II

Connes' Integral Formula (1988)

Let (M, g) be a d-dimensional compact orientable Riemannian manifold. Then for any Dixmier trace Tr_ω,

$$\mathrm{Tr}_\omega(M_f(1-\Delta_g)^{-\frac{d}{2}}) = C_d \int_M f \, d\nu_g, \quad f \in C(M),$$

where Δ_g is the Laplace–Beltrami operator, ν_g the Riemannian volume form, and C_d a constant depending on the dimension d.

Part 1: 1915

Toeplitz Matrices

Let $\{e_n\}_{n\in\mathbb{Z}}$ be the standard basis of $L_2(\mathbb{S}^1)$. The matrix elements of the multiplication operator M_w are

$$\langle e_n, M_w e_m\rangle = \frac{1}{2\pi}\int_0^{\theta} e^{-in\theta} w(\theta) e^{im\theta}\, d\theta = w_{m-n}.$$

The matrix elements of T_n are also of the form w_{m-n}.

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The matrix elements of T_n are also of the form w_{m-n}.

We therefore have

$$T_n = P_n M_w P_n,$$

where where P_n is the orthogonal projection in $L_2(\mathbb{S}^1)$ onto the Fourier modes $\{e_0, e_1, \ldots, e_n\}$.

Reworking Szegő's theorem

Szegő's limit theorem gives

$$\lim_{n\to\infty} (\det T_n)^{\frac{1}{n}} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log(w(\theta))\, d\theta\right).$$

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We have

$$\log(\det(T)) = \log\left(\prod_{\lambda_j\in\sigma(T)}\lambda_j\right) = \sum_{\lambda_j\in\sigma(T)}\log(\lambda_j) = \mathrm{Tr}(\log(T)),$$

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hence a different way to put Szegő's theorem is

$$\frac{1}{n+1} \mathrm{Tr}(\log(P_n M_w P_n)) \xrightarrow{n\to\infty} \int_{\mathbb{S}^1} \log(w(\theta))\, d\nu(\theta),$$

where P_n is the orthogonal projection in $L_2(\mathbb{S}^1)$ onto the Fourier modes $\{e_0, e_1, \ldots, e_n\}$.

Szegő's even better limit theorem

In fact, Szegő proved a stronger statement.

Szegő's even better limit theorem (1915)

For $w \in C(\mathbb{S}^1)$ real-valued,

$$\frac{1}{n+1}\mathrm{Tr}(f(P_n M_w P_n)) \xrightarrow{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(w(\theta))\, d\theta, \quad f \in C(\mathbb{R}).$$

Part 2: 1979

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Widom's Szegő's limit theorem (1979)

Let (M, g) be a compact Riemannian manifold, $w \in C(M)$ real-valued. Then

$$\frac{\mathrm{Tr}(f(P_\lambda M_w P_\lambda))}{\mathrm{Tr}(P_\lambda)} \xrightarrow{\lambda\to\infty} \int_M f(w(x))\, d\nu_g(x), \quad f \in C(\mathbb{R}),$$

where $P_\lambda = \chi_{[-\lambda,\lambda]}(\Delta_g)$.

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where $P_\lambda = \chi_{[-\lambda,\lambda]}(\Delta_g)$.

Note: Widom only claimed his result for tori, but his arguments can easily be used for all compact Riemannian manifolds. Also he gave a microlocal version.

Idea

Widom's proof is quite short.

- The first step is to prove the case where $f(x) = x$, i.e.

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- The next step is to do polynomials, by proving that

$$\frac{\mathrm{Tr}((P_\lambda M_w P_\lambda)^n - P_\lambda M_w^n P_\lambda)}{\mathrm{Tr}(P_\lambda)} \xrightarrow{\lambda\to\infty} 0.$$

Part 3: 1988

Dixmier traces

Let $\mathcal{H}$ be a Hilbert space. An eigenvalue sequence of a compact operator $A \in K(\mathcal{H})$ is a sequence $\{\lambda(k, A)\}_{k\in\mathbb{N}}$ of the eigenvalues of A listed with multiplicity, such that $\{|\lambda(k, A)|\}_{k\in\mathbb{N}}$ is non-increasing.

The usual operator trace Tr can be characterised for trace class operators $A \in \mathcal{L}_1 \subset K(\mathcal{H})$ as

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The Dixmier trace is defined on so-called weak trace class operators $A \in \mathcal{L}_{1,\infty} \subset K(\mathcal{H})$ by

$$\mathrm{Tr}_\omega(A) = \omega\text{-}\lim_{n\to\infty} \frac{1}{\log(2+n)} \sum_{k=1}^{n} \lambda(k, A),$$

where $\omega \in \ell_\infty(\mathbb{N})^*$ is an extended limit. Note that $\mathcal{L}_1 \subset \mathcal{L}_{1,\infty}$, but if $A \in \mathcal{L}_1$, $\mathrm{Tr}_\omega(A) = 0$.

Connes' integral formula

Connes proved the following.

Connes' Integral Formula

Let (M, g) be a compact Riemannian manifold, $f \in C_c(M)$. Then for any Dixmier trace Tr_ω,

$$\mathrm{Tr}_\omega(M_f(1 - \Delta_g)^{-\frac{d}{2}}) = C_d \int_M f \, d\nu_g.$$

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Also here, Connes actually gave a microlocal version. Even more, he gave a version without assuming a Riemannian structure.

Part 4: 2025 (I)

Comparison

Now compare the first step in Widom's proof, the formula

$$\frac{\mathrm{Tr}(P_\lambda M_w P_\lambda)}{\mathrm{Tr}(P_\lambda)} \xrightarrow{\lambda\to\infty} \int_M w(x)\, d\nu_g(x),$$

with Connes' formula

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$$\mathrm{Tr}_\omega(M_w(1-\Delta)^{-\frac{d}{2}}) = \int_M w(x)\, d\nu_g(x).$$

H.–McDonald

Let $\mathcal{H}$ be a separable Hilbert space, $A \in B(\mathcal{H})$, D self-adjoint with compact resolvent, $P_\lambda := \chi_{[-\lambda,\lambda]}(D)$. If D satisfies Weyl's law $\lambda(k,|D|) \sim Ck^{\frac{1}{d}}$, then for all Dixmier traces Tr_ω,

$$\frac{\mathrm{Tr}_\omega(A(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})} = \omega \circ M\left(\frac{\mathrm{Tr}(P_{\lambda_n} A P_{\lambda_n})}{\mathrm{Tr}(P_{\lambda_n})}\right).$$

Here, $M : \ell_\infty \to \ell_\infty$ is the logarithmc averaging defined by $M(x)_n = \frac{1}{\log(n+2)} \sum_{k=0}^{n} \frac{x_k}{k}$.

Truncated Spectral Triples

If we have a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, lots of noncommutative geometers are interested in truncated triples $(P_\lambda A P_\lambda, P_\lambda \mathcal{H}, P_\lambda D)$ (e.g. Connes–van Suijlekom, D'Andrea–Lizzi–Martinetti).

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Our result shows that if $(\mathcal{A}, \mathcal{H}, D)$ is d-dimensional and D satisfies Weyl's law, then

$$P_\lambda A P_\lambda \mapsto \frac{\mathrm{Tr}(P_\lambda A P_\lambda)}{\mathrm{Tr}(P_\lambda)}$$

is a reasonable approximation of the noncommutative integral $\mathrm{Tr}_\omega(A(1 + D^2)^{-\frac{d}{2}})$.

Noncommutative Szegő theorem

With Widom's argument, we have a Szegő formula for NCG as well.

H.–McDonald

Let $\mathcal{H}$ be a separable Hilbert space, $A \in B(\mathcal{H})_{sa}$, D self-adjoint with compact resolvent. If D satisfies Weyl's law $\lambda(k, |D|) \sim Ck^{\frac{1}{d}}$, and if $[D, A]$ extends to a bounded operator, then for all Dixmier traces Tr_ω,

$$\frac{\mathrm{Tr}_\omega(f(A)(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})} = \omega \circ M\left(\frac{\mathrm{Tr}(f(P_{\lambda_n} M_w P_{\lambda_n}))}{\mathrm{Tr}(P_{\lambda_n})}\right), \quad f \in C_c(\mathbb{R}).$$

Part 5: 2025 (II)

Quantum Ergodicity

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This field studies a quantum mechanical analogue of ergodicity. A one-particle system described by an operator H on $L_2(M)$ is called quantum ergodic if the high energy states of H are 'smeared' over M.

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Quantum Ergodicity

For a compact Riemannian manifold M and a positive self-adjoint operator Δ on $L_2(M)$ with compact resolvent, Δ is said to be quantum ergodic if for every orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of $L_2(M)$ consisting of eigenfunctions of Δ, there exists a density one subsequence $J \subseteq \mathbb{N}$ such that

$$\lim_{J \ni j \to \infty} \langle e_j, M_f e_j \rangle_{L_2(M)} \to \int_M f \, d\nu, \quad f \in C(M),$$

where ν is a probability measure on M. In this context, a density one subsequence means that

$$\frac{\#(J \cap \{0, \ldots, n\})}{n+1} \to 1, \quad n \to \infty.$$

Picture

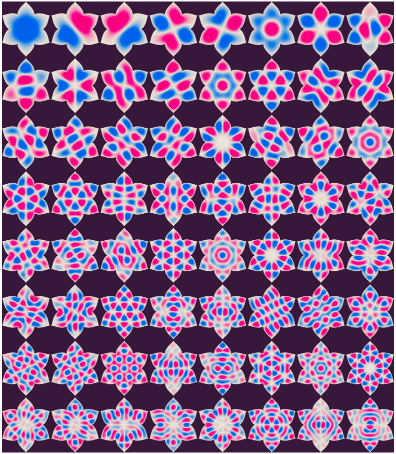

Figure: Eigenfunctions of the Laplacian on a rose-shaped domain, *probably* quantum ergodic.

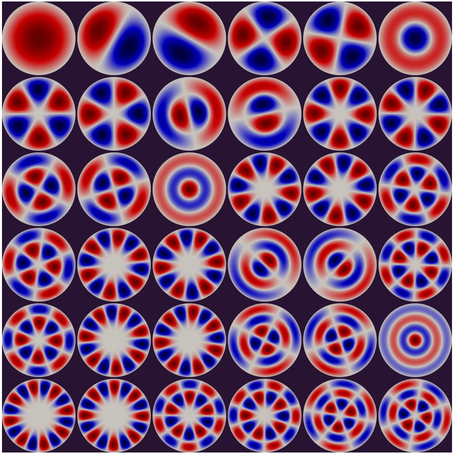

Figure: Eigenfunctions of the Laplacian on the disc, *not* quantum ergodic.

Geodesic flow

Let (M, g) be a Riemannian manifold, and denote by $SM \subseteq TM$ the tangent vectors of length 1. Then we define the geodesic flow

$$G_t : SM \to SM, \quad t \in \mathbb{R},$$

in the usual way. By duality, we can likewise define $G_t : S^*M \to S^*M$, where $S^*M \subseteq T^*M$.

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The geodesic flow is said to be *ergodic* if for every measurable function $f \in L_\infty(S^*M)$ which is fixed by the flow (i.e. $f \circ G_t = f$ almost everywhere), it must be that f is constant almost everywhere.

More pictures

Fundamental theorem of QE

The fundamental theorem that started the field of Quantum Ergodicity is the following.

Theorem (Shnirelman 1974, Zelditch 1987, Colin de Verdière 1985)

Let M be a compact Riemannian manifold. If the geodesic flow on M is ergodic, then the Laplace–Beltrami operator Δ_g is quantum ergodic.

Fundamental theorem of QE

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Theorem (Shnirelman 1974, Zelditch 1987, Colin de Verdière 1985)

Let M be a compact Riemannian manifold. If the geodesic flow on M is ergodic, then the Laplace–Beltrami operator Δ_g is quantum ergodic.

By now, various extensions of this theorem exist. The common thread is to study geodesic flow, and translate this into asymptotic behaviour of eigenfunctions of an operator.

An NCG perspective

To show QE of an operator Δ, the goal is to show that $\langle e_j, M_f e_j\rangle$ converges to $\int_M f\, d\nu$ in some sense, but often the starting point is that

$$\frac{1}{N+1}\sum_{j=0}^{N}\langle e_j, M_f e_j\rangle = \frac{1}{N+1}\mathrm{Tr}(Q_N M_f Q_N) \xrightarrow{N\to\infty} \int_M f\, d\nu,$$

where Q_N is the projection onto the first $N+1$ eigenvectors of Δ.

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where Q_N is the projection onto the first $N+1$ eigenvectors of Δ.

A link with our previous result,

$$\omega\circ M\left(\frac{\mathrm{Tr}(P_{\lambda_n} A P_{\lambda_n})}{\mathrm{Tr}(P_{\lambda_n})}\right) = \frac{\mathrm{Tr}_\omega(A(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})},$$

is apparent. We have this step down, but what is the NCG analogue of ergodic geodesic flow?

Noncommutative geodesic flow

Connes defined geodesic flow on spectral triples in 1996.

Noncommutative cosphere bundle

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple. Let $\sigma_t : B(\mathcal{H}) \to B(\mathcal{H})$ be defined by $A \mapsto e^{it|D|}Ae^{-it|D|}$. Then define the C^*-algebra

$$S^*\mathcal{A} := C^*\left(\bigcup_{t\in\mathbb{R}} \sigma_t(\mathcal{A}) + K(\mathcal{H})\right)\Big/ K(\mathcal{H}),$$

where σ_t descends to an action of $\mathbb{R}$ on $S^*\mathcal{A}$. If $(\mathcal{A}, \mathcal{H}, D) \simeq (C^\infty(M), L_2(S), D_S)$, then $S^*\mathcal{A} \simeq C(S^*M)$ and σ_t is the geodesic flow.

NCG ergodicity

Since ergodicity of the geodesic flow is a measure theoretic statement, we need to take one more step.

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L_2-cosphere bundle (H.–McDonald)

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple where D satisfies Weyl's law. Then

$$\tau(A + K(\mathcal{H})) = \frac{\mathrm{Tr}_\omega(A(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})}, \quad A + K(\mathcal{H}) \in S^*\mathcal{A},$$

defines a finite positive trace on $S^*\mathcal{A}$. Then define $L_2(S^*\mathcal{A})$ as the Hilbert space of the GNS representation of $S^*\mathcal{A}$ corresponding to τ.
The geodesic flow σ_t on $S^*\mathcal{A}$ descends to a unitary operator on $L_2(S^*\mathcal{A})$.

NCG QE

We can now naively put forward a definition of ergodic geodesic flow for spectral triples. Namely, we say that the geodesic flow σ_t is ergodic on $(\mathcal{A}, \mathcal{H}, D)$ if the only σ_t-invariant element of $L_2(S^*\mathcal{A})$ is the identity.

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NCG QE (H.–McDonald)

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple where D satisfies Weyl's law. If the geodesic flow on $(\mathcal{A}, \mathcal{H}, D)$ is ergodic, then D is quantum ergodic. That is, for every basis $\{e_n\}_{n=0}^{\infty}$ of $\mathcal{H}$ consisting of eigenvectors of D, there exists a density one subset $J \subseteq \mathbb{N}$ such that

$$\lim_{J \ni j \to \infty} \langle e_j, a e_j \rangle = \frac{\mathrm{Tr}_\omega(a(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})}.$$

Future directions

There are some questions left over.

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- A recent QE paper by Ma and Ma proved the QE property in a setting with vector bundles, which can be seen through our NCG lens. Can we check the QE property for some interesting noncommutative geometries?
- These questions are linked: if a space is quantum ergodic, then a subset (with density 1) of the matrix elements $\langle e_j, ae_j \rangle$ will converge to the noncommutative integral of a. This might suggest a faster convergence of $\frac{\mathrm{Tr}(P_\lambda a P_\lambda)}{\mathrm{Tr}(P_\lambda)}$ to the noncommutative integral of a.

Thanks

Thanks for listening!